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# ON THE PROPERTIES OF THE ENDOMORPHISMS OF RESIDUALLY FINITE GROUPS

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**Abstract :**

R. Hirshon showed in [3] that : if $\alpha$ is an endomorphism of a finitely generated residually finite group G and the subgroup $G\alpha$ has a finite index in G, then there exists a positive integer k such that $\alpha$ induces an isomorphism on $G\alpha^k$.

The group $G_{mn} =< a,b;[a^m,b^n]=1>$ m, n > 1 is finitely generated, residually finite and any of its endomorphism $\varphi$ satisfies the conclusion of the Hirshon's theorem stated above without any restriction on the index of the subgroup $G_{mn}\varphi$.

**Keywords :**

Finitely generated residually finite groups. Free products of groups with amalgamations. H. Neumann's theorem. Tietze transformations.

### 0. Introduction.

R. Hirshon showed in his paper [3] that: if $\alpha$ is an endomorphism of a finitely generated residually finite group G and the subgroup $G\alpha$ has a finite index in G, then there exists a positive integer k such that $\alpha$ induces isomorphism on $G\alpha^k$. He also posed the question whether any endomorphism $\alpha$ of a finitely generated residually finite group G satisfies the conclusion of the Hirshon's theorem stated above, without any restriction on the index of the subgroup $G\alpha$.

If G is a free group of finite rank, this question has a positive answer. It is also true for the groups $\langle a,b\ ;\ a^{-1}ba = b^k\rangle$, $k \neq 0$ and $\langle a,b\ ;\ a^m = b^n\rangle$, m,n > 1. See [1] and [2] respectively.

The endomorphisms of the group $G_{mn} =< a, b;[a^m,b^n]=1>$ m,n > 1 were characterised in [6].

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The purpose of this paper is to show that any endomorphism $\varphi$ of the group $G_{mn} =< a, b;[a^m,b^n]=1>$, where m and n are integers greater than 1, satisfies the conclusion of the Hirshon's theorem stated above, without any restriction on the index of the subgroup $G_{mn}\varphi$.

Take for example m = 2, n = 3 i.e. $G_{23} =< a, b;[a^2,b^3]=1>$ and $\varphi$ as follows : $a\varphi = u = a^3$, $b\varphi = v = b^6$. Then $G_{23}\varphi =< u, v;[u^2,v]=1>$, $G_{23}\varphi^2 =< u',v';[(u')^2,v']=1>$, where $u' = a^9$, $v' = b^{36}$. We then have $G_{23}\varphi \cong G_{23}\varphi^2$ by $u\varphi = u'$ and $v\varphi = v'$ while $[G_{23}:G_{23}\varphi]=\infty$.

### 1. Preliminaries and properties of the group $G_{mn}$.

Let $H=< c,d;\ cd = dc >$ be the free abelian group of rank 2; let $A=(< a >*H;\ a^m = c)$ be the free product of $< a >$, the free group generated by the element a and the free abelian group H, amalgamated by $< a^m = c >$, the subgroup generated by the element $a^m = c$; let $B=(H*< b >;\ d = b^n)$ be the free product of the free abelian group H and $< b >$, the free group generated by the element b, amalgamated by $< b^m = d >$, the subgroup generated by the element $b^n = d$. Then by Tietze transformations[2],

$$A =< a,c,d;\ a^m = c,\ cd = dc >,$$

$$B =< c,d,b;\ b^n = d,\ cd = dc >$$

and $G_{mn} = (A*B;H)$, the free product of groups A and B, amalgamated by the subgroup H; this means that $G_{mn}$ is generated by the groups A and B, where $A \cap B = H$ and any element g of $G_{mn}$ has the form :

$$g = g_1 g_2 \cdots g_n \qquad \text{(I)}$$

where each component $g_i$ $(i=1,2,\dots,n)$ belongs to one of the subgroup A or B and for n > 1, consecutive components are not both in one of the subgroup A or B.

The form (I) with these conditions is called the **reduced form** of the element g and then g is **irreducible**.

$f_1 f_2 \cdots f_r$ and $g_1 g_2 \cdots g_s$ are reduced forms of the same element g of $G_{mn}$ iff r = s and there exist elements $h_1, h_2, \dots, h_{r-1}$ of the amalgamated subgroup H such that :

[2] For details about Tietze transformations, see [4] or [5].

$$\begin{cases} f_1 = g_1h_1 \\ f_i = (h_{i-1})^{-1}g_ih_i,\ (1<i<r). \\ f_r = (h_{r-1})^{-1}g_r \end{cases}$$

So, the reduced forms are not unique for a given element. But the number of the components is the same for all these forms. This number is called the **length** of the element g or the **length of the reduced form** of g and will be denoted **l(g)**. An element g of the group $G_{mn}$ is said to be **cyclically irreducible** if for its reduced form $g = g_1g_2\cdots g_n$, either n = 1, or n > 1 and the components $g_1$ and $g_n$ are not both in one of the subgroup A or B.

If an element g of $G_{mn}$ is not cyclically irreducible, then it can be conjugated to a cyclically irreducible element, that is :

$$g = x_1x_2\cdots x_r(y_1y_2\cdots y_s)x_r^{-1}x_{r-1}^{-1}\cdots x_1^{-1}, \qquad \text{(II)}$$

where $r \geq 1$, $s \geq 1$, $x_1x_2\cdots x_r$ is reduced, $y_1y_2\cdots y_s$ - cyclically reduced and components $x_r$ and $y_s$ are not both in one of the subgroup A or B, and for s > 1, $y_sx_r^{-1} \notin H$.

We say that an element g is **transformed** if s = 1 in the script (II).

Details of the above results can be found in [4] or [5].

We now prove some useful properties of amalgamated free product.

Let P = (X∗ Y;U) be the free product of groups X and Y amalgamated by the subgroup U.

**Proposition 1.1** (H. Neumann's theorem).

Let X' and Y' be subgroups of X and Y respectively, such that $X'\cap U = Y'\cap U = U'$. Then $P' = sgp(X', Y')$, the group generated by the subgroups X' and Y' is the free product of X' and Y' amalgamated by the subgroup U'; that is $P' = (X' * Y'; U')$ and furthermore $P'\cap X = X'$, $P'\cap Y = Y'$.

For the proof, see [5] p. 512. □

**Proposition 1.2** If an element g of P belongs to the centralizer of the subgroup U, then $P_g = sgp(g^{-1}Xg, Y)$, the group generated by subgroups $g^{-1}Xg$ and Y is the free product of these subgroups amalgamated by U; that is $P_g = ((g^{-1}Xg) * Y; U)$.





**Proof.** It suffices to show that $(g^{-1}Xg)\cap Y = U$ if g is in the centralizer of U, and apply proposition 1.1.

So we have $U\subseteq X$ and $U \subseteq g^{-1}Ug$ as g centralises U; then $U \subseteq g^{-1}Ug \subseteq g^{-1}Xg$. Also $U\subseteq Y$, thus $U \subseteq g^{-1}Xg \cap Y$. Conversely, let $g^{-1}xg \in Y$, where $x\in X\setminus U$ and g centralises U. If $l(g) > 1$, then $g^{-1}xg \in Y$ iff $x\in U$ for if not $l(g^{-1}xg) > 1$ and $g^{-1}xg$ cannot belong to Y. So $g^{-1}xg = x \in U \subseteq Y$ and $g^{-1}Xg\cap Y\subseteq U$. □

**Theorem 1.3** Any endomorphism φ of the group $G_{mn}$, up to inner automorphisms, has the form aφ = u, bφ = v, where either one of the following properties holds :

(1) u and v are cyclically irreducible elements of lengths greater than 1 and uv = vu.

(2) u and v belong to the same subgroup A (or B) and $u^mv^n = v^nu^m$.

(3) $u = x^{-1}a^kx$, $v = y^{-1}b^ly$, where $x \in A$, $y \in B$ and $k, l \in Z$.

(4) $u = y^{-1}b^ky$, $v = x^{-1}a^lx$, where $x \in A$, $y \in B$ and $k, l \in Z$, n divides km and m divides nl.

This theorem is proved in [6]. □

## 2. Endomorphisms of and Hirshon's theorem.

**Theorem 2.1** Any endomorphism φ of type (1) in theorem 1.3 satisfies the conclusion of the Hirshon's theorem without any restriction on the index of the subgroup $G_{mn}\varphi$.

**Proof.** Since u and v are cyclically irreducible with lengths greater than 1 and uv = vu, then $G_{mn}\varphi$ = sgp(u,v) = <u,v; uv = vu> is finitely generated, abelian and has a rank which is not greater than 2 and the equality of ranks of the subgroups $G_{mn}\varphi^i$ and $G_{mn}\varphi^{i+1}$ means that the map $\varphi : G_{mn}\varphi^i \to G_{mn}\varphi^{i+1}$ is one to one. Since the sequence of ranks of the subgroups $G_{mn}\varphi^i$ (i = 1, 2, ...) is stationary, then this proves the Hirshon's theorem without any restriction on the index of the subgroup. □

We now examine endomorphisms of type (3) in theorem 1.3.

Let φ be such an endomorphism. For any i = 1, 2,... the endomorphism $\varphi^i$ is defined by :

$a\varphi^i = x_i' = x_i^{-1}a^{k^i}x_i$, $b\varphi^i = y_i' = y_i^{-1}b^{l^i}y_i$, where $x_i = x \cdot x\varphi \cdot x\varphi^2 \cdots x\varphi^{i-1}$ and $y_i = y \cdot y\varphi \cdot y\varphi^2 \cdots y\varphi^{i-1}$. In particular, if $x = y = 1$, we obtain the endomorphism $\varphi_{kl}$ so defined : $a\varphi_{kl} = a^k$, $b\varphi_{kl} = b^l$, where $k, l \in Z$.

So for any $i = 1, 2, \ldots$ we have : $a\varphi_{kl}^i = a^{k^i}$, $b\varphi_{kl}^i = b^{l^i}$.

Now let $\begin{cases} t_i = \gcd(k^i, m), \text{ that is } k^i = k_i t_i \text{ and } m = m_i t_i \\ t_i' = \gcd(l^i, n), \text{ that is } l^i = l_i t_i' \text{ and } n = n_i t_i' \end{cases}$ (III),

where gcd(x, y) is the greatest common divisor of the integers x and y.

Let $c_i = c^{k_i}$ and $d_i = d^{l_i}$.

Let $H_i = sgp(c_i, d_i)$ and $A_i = sgp(a^{k^i}, H_i)$. By Proposition 1.1, $A_i = (<a^{k^i}> * H_i; (a^{k^i})^{m_i} = c_i)$ and $B_i = sgp(b^{l^i}, H_i) = (<b^{l^i}> * H_i; (b^{l^i})^{n_i} = d_i)$. So if, $G_i = sgp(A_i, B_i)$, then $G_i = (A_i * B_i; H_i)$ and by Tietze transformations, we have : $G_i = <a^{k^i}, b^{l^i}; [(a^{k^i})^{m_i}, (b^{l^i})^{n_i}] = 1>$ with $m_i$, $n_i$ as in (III).

Let's examine $G_{x_i y_i} = sgp(x_i', y_i')$; $A_i' = sgp(x_i^{-1}ax_i, H) = (<x_i^{-1}ax_i> * H; (x_i^{-1}ax_i)^m = c)$ by proposition 1.2. It is obvious that $A_i' \cong A$ by the isomorphism : $x_i^{-1}ax_i \mapsto a, c \mapsto c, d \mapsto d$.

Similarly, $B_i' = sgp(y_i^{-1}by_i, H) = (<y_i^{-1}by_i> * H; (y_i^{-1}by_i)^n = d)$ and $B_i' \cong B$ by the isomorphism : $y_i^{-1}by_i \mapsto b, c \mapsto c, d \mapsto d$. So if $G_i' = sgp(A_i', B_i')$, then $G_i' = (A_i' * B_i'; H)$ and $G_i' \cong G_{mn}$ by the map $\theta$ defined by : $x_i^{-1}ax_i \mapsto a, y_i^{-1}by_i \mapsto b$. Obviously, $x_i'\theta = a^{k^i}$ and $y_i'\theta = b^{l^i}$; so $G_{x_i y_i}\theta = G_i$ and we have the following proposition.

**Proposition 2.2** $G_{x_i y_i} = <x_i', y_i'; [(x_i')^{m_i}, (y_i')^{n_i}] = 1>$ with $m_i$, $n_i$ as defined in (III). □

**Theorem 2.3** Any endomorphism $\varphi$ of type (3) (respectively $\psi$ of type (4)) in theorem 1.3 satisfies the conclusion of the Hirshon's theorem without any restriction on the index of the subgroup $G_{mn}\varphi$ (respectively $G_{mn}\psi$).

**Proof.** We first note that the sequences $(t_i)$ and $(t_i')$ ($i \in$ IN) defined in (III) are stationary since m and n are fixed constants. So there exists a rank r from which the sequences $(t_i)$ and $(t_i')$ are both stationary : this means that,



$t_r = \gcd(k^r, m) = \gcd(k^{r+1}, m) = t_{r+1}$, that is $m = t_r m_r = t_{r+1} m_{r+1}$. So $m_r = m_{r+1}$ as $t_r = t_{r+1}$. Similarly $n_r = n_{r+1}$. So far, by proposition 2.2 we have : $G_{mn}\varphi^r = <x_r', y_r'; [(x_r')^{m_r}, (y_r')^{n_r}] = 1>$ and $G_{mn}\varphi^{r+1} = <x_{r+1}', y_{r+1}'; [(x_{r+1}')^{m_{r+1}}, (y_{r+1}')^{n_{r+1}}] = 1> = <x_{r+1}', y_{r+1}'; [(x_{r+1}')^{m_r}, (y_{r+1}')^{n_r}] = 1>$.

As $x_r'\varphi = x_{r+1}'$ and $y_r'\varphi = y_{r+1}'$, then $G_{mn}\varphi^r \cong G_{mn}\varphi^{r+1}$ and $\varphi$ induces an isomorphism on the subgroups $G_{mn}\varphi^r$ and $G_{mn}\varphi^{r+1}$.

As $A\psi \subseteq B$ and $B\psi \subseteq A$, then for $i = 1, 2, \ldots$, $\psi^i$ is an endomorphism of type (3) when i is even and of type (4) when i is odd. So, as above, there is a rank r from which the endomorphism $\psi^2$ induces an isomorphism from the subgroup $G_{mn}\varphi^{2r}$ to $G_{mn}\varphi^{2r+2}$; then $\psi$ induces an isomorphism from $G_{mn}\varphi^{2r}$ to $G_{mn}\varphi^{2r+2}$ and the theorem is proved. □

Let's now examine endomorphisms of type (2) in theorem 1.3.

Let $\varphi$ be such an endomorphism. We have $a\varphi = u$, $b\varphi = v$, where $u, v \in A$ (or B) and $u^m v^n = v^n u^m$. For the sake of simplicity,

let $u, v \in A = sgp(a, c, d) = <a,c,d; a^m = c, cd = dc> = (<a> * H; H')$,

where $H' = <a^m = c>$, the case $u, v \in B$ being similar.

**Proposition 2.4** Let $v \in A \backslash <a>$ (or $A \backslash H$) and $v^k \in <a>$ (or H respectively) for an integer $k > 1$. Then $v = wa^t w^{-1}$, where $w \in A$, $t \in Z$, m does not divide t and m divides kt.

**Proof.** Let $v \in A \backslash <a>$ and $v^k \in <a>$. Then v is transform in A; that is $v = x_1 x_2 \cdots x_r y x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1}$, where $x_1 x_2 \cdots x_r$ is irreducible in A, $y \in <a> \backslash H'$ (or $H \backslash H'$) and $y^k \in H'$.

If $r = 0$, as $v \notin <a>$, then $v = y \in H \backslash H'$ and $v^k \notin <a>$; so $r > 0$; and if $y \in H \backslash H'$, then $v^k \notin <a>$; so $y \in <a> \backslash H'$, that is $y = a^t$, m does not divide t and $v^k = x_1 x_2 \cdots x_r a^{tk} x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1} = x_1 x_2 \cdots x_r c^q x_r^{-1} x_{r-1}^{-1} \cdots x_1^{-1}$ where $kt = mq$; thus we take $w = x_1 x_2 \cdots x_r$.

Similarly one proves the case $v \in A \backslash H$ and $v^k \in H$, for an integer $k > 1$ and the proposition is proved. □

**Proposition 2.5** Any endomorphism $\varphi$ of type (2) in theorem 1.3, up to inner automorphisms, has the form $a\varphi = u$, $b\varphi = v$, where either one of the following properties holds :

(a) u and v are cyclically irreducible and $uv = vu$,

(b) $u \in A$ and $v = wa^t w^{-1}$, where $w \in A$, $t \in Z$, and m divides nt,

(c) $u \in \langle a \rangle$ and $v \in A$.

**Proof.** Similar to the proof of theorem 1.3. □

**Remark.** As in theorem 2.1, any endomorphism φ of type (a) in the proposition 2.5 induces an isomorphism of subgroups without any restriction on the index of the subgroup $G_{mn}\varphi$, if $l(u), l(v) > 1$.

For $l(u) = l(v) = 1$, see the proof (1st case) of theorem 2.6 below.

We now turn to endomorphisms of type (b) and (c) in the proposition 2.5.

**Theorem 2.6** Endomorphisms of type (b) and (c) in proposition 2.5 satisfy the conclusion of the Hirshon's theorem without any restriction on the index of subgroups, homomorphic images of the group $G_{mn}$.

**Proof.** Let φ be an endomorphism of type (b) : then $a\varphi = u$, $b\varphi = v = wa^t w^{-1}$, where $u, w \in A$, $t \in Z$, and m divides nt. It is obvious by induction on i that : for any $z \in A$, $z\varphi^i = u^{p_i}c^{q_i}$, for $i = 1, 2, \ldots$, $a\varphi^i = u^{r_i}c^{s_i}$, $b\varphi^i = u^{r_i'}c^{s_i'}$, for $i = 2, 3, \ldots$, where $p_i, q_i, r_i, s_i, r_i'$ and $s_i'$ are all integers depending on i.

Two cases arise depending on whether $u^{r_i}$ and $u^{r_i'}$ are both in $\langle c \rangle$ or not.

<u>1st case</u> : If $u^{r_i}$ and $u^{r_i'}$ are in $\langle c \rangle$, then $sgp(u^{r_i}c^{s_i}, u^{r_i'}c^{s_i'}) = sgp(c^{p(i)}, c^{q(i)}) = \langle c^{j(i)} \rangle$, where $j(i) = gcd(p(i), q(i))$, $p(i), q(i) \in Z$. So, from $i = 2$, $G_{mn}\varphi^i = \langle c^{j(i)} \rangle$ and these subgroups $G_{mn}\varphi^i$ are isomorphic, as infinite cyclic groups.

<u>2nd case</u> : If at least one of the elements $u^{r_i}$ or $u^{r_i'}$ does not belong to $\langle c \rangle$, then

$$sgp(u^{r_i}c^{s_i}, u^{r_i'}c^{s_i'}) = \langle u^{r_i}, c^{s_i}, u^{r_i'}, c^{s_i'}; u^{r_i}c^{s_i} = c^{s_i}u^{r_i}, u^{r_i'}c^{s_i'} = c^{s_i'}u^{r_i'} \rangle$$

$$= \langle u^{f(i)}, c^{g(i)}; u^{f(i)}c^{g(i)} = c^{g(i)}u^{f(i)} \rangle \text{ as c centralises A,}$$

where $f(i) = gcd(r_i, r_i')$ and $g(i) = gcd(s_i, s_i')$.

Then for $i = 2, 3, \ldots$, the subgroups $G_{mn}\varphi^i$ are subgroups of the free abelian group of rank less than or equal to 2 and hence are isomorphic by the proof of theorem 2.1. The conclusion of the Hirshon's theorem holds without any restriction on the index of the subgroup $G_{mn}\varphi$.





Let now ψ be an endomorphism of type (c); that is $a\psi = a^p$, $p \in Z$, $b\psi = v \in A$.

For any $i = 1, 2, \ldots$, $a\psi^i = a^{p^i}$, $b\psi^i = v_i$, where $v_1 = b\psi = v$ and $v_i = v\psi^{i-1}$, $(i > 1)$. Since $v_i$ is any element of A, by Tietze transformations, $sgp(a^{p^i}, v_i) = \langle a^{p^i}, v_i; [(a^{p^i})^{m_i}, v_i] = 1 \rangle$, where $m = (gcd(a^{p^i}, m)) \cdot m_i$. m is a fixed integer or a constant, so the sequence $(m_i)$ $i = 1, 2, \ldots$, is stationary; thus, as in the proof of theorem 2.3, the endomorphism ψ induces isomorphism from a certain rank. □

**Conclusion.**

We have shown in this paper that any endomorphism φ of the group $G_{mn} = \langle a, b; [a^m, b^n] = 1 \rangle$, $m, n > 1$ satisfies the conclusion of the Hirshon's theorem without any restriction on the index of the subgroup $G_{mn}\varphi$. But the following question posed by Hirshon is still open : "If α is an endomorphism of a finitely generated residually finite group G, does there exist a positive integer k such that α induces an isomorphism on $G\alpha^k$ ?"

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